

CS-233 Theoretical Exercise 7

February 2024

1 Valid Kernels

Let us assume that a kernel function $k(\mathbf{x}_i, \mathbf{x}_j)$ is valid if there exists a mapping $\phi(\cdot)$ to the real domain such that $k(\mathbf{x}_i, \mathbf{x}_j) = \phi(\mathbf{x}_i)^T \phi(\mathbf{x}_j)$ for all \mathbf{x}_i and \mathbf{x}_j .

Question 1: Decide whether the following are valid kernels and explain why.

1. $k(\mathbf{x}_i, \mathbf{x}_j) = 4$
2. $k(\mathbf{x}_i, \mathbf{x}_j) = -4$
3. $k(\mathbf{x}_i, \mathbf{x}_j) = (\mathbf{x}_i^T \mathbf{x}_j)^2$
4. $k(\mathbf{x}_i, \mathbf{x}_j) = \cos(\mathbf{x}_i - \mathbf{x}_j)$
5. $k(\mathbf{x}_i, \mathbf{x}_j) = \cos(\mathbf{x}_i) \sin(\mathbf{x}_j)$

2 Constructing Kernels

One powerful technique for constructing new kernels is to build them out of simpler kernels as building blocks. Assume that kernels $k_1(\mathbf{x}_i, \mathbf{x}_j)$ and $k_2(\mathbf{x}_i, \mathbf{x}_j)$ are valid.

Question 2: Show that the following kernels are valid.

1. $k(\mathbf{x}_i, \mathbf{x}_j) = ck_1(\mathbf{x}_i, \mathbf{x}_j)$, where $c > 0$ is a constant
2. $k(\mathbf{x}_i, \mathbf{x}_j) = k_1(\mathbf{x}_i, \mathbf{x}_j) + k_2(\mathbf{x}_i, \mathbf{x}_j)$
3. $k(\mathbf{x}_i, \mathbf{x}_j) = k_1(\mathbf{x}_i, \mathbf{x}_j)k_2(\mathbf{x}_i, \mathbf{x}_j)$

3 Kernelized Nearest Neighbor

The standard version of the Nearest Neighbor algorithm exploits the Euclidean distance $d(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|$. If we have a valid kernel $k(\mathbf{x}_i, \mathbf{x}_j)$, then we can define a new distance based on the kernel as $d_k(\mathbf{x}_i, \mathbf{x}_j) = \sqrt{k(\mathbf{x}_i - \mathbf{x}_j, \mathbf{x}_i - \mathbf{x}_j)}$. Suppose that we have three points on a plane $(0, 1), (1, 0), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

Question 3: Find the nearest neighbor of $(0, 0)$ under the following distances based on different kernels.

1. $k(\mathbf{x}_i, \mathbf{x}_j) = x_i^T x_j$
2. $k(\mathbf{x}_i, \mathbf{x}_j) = x_i^T \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} x_j$